The Electron and the Muon as Eigensolutions of the Quantum Electromechanics under the Green-Function $\delta(x^2 + l^2)$ *

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Replacing the Green function of Maxwell's electrodynamics $\delta(x^2)$ by $\delta(x^2+l^2)$ we obtain a Hamiltonian with a *finite* number of degrees of freedom for the classical motion of a pointcharge in its own electromagnetic field. After quantization we obtain a mass spectrum if we assume that a nonelectrodynamic bare mass M exists. The spectral terms are $S_{1/2}$, $P_{1/2}$; $P_{3/2}$, $D_{3/2}$; $D_{5/2}$ etc. ($k=+1, -1; +2, -2; +3 \ldots$). It is possible to fit the length l in the Green function and the mass M so that the mass ratio of the lowest terms becomes $m(P_{1/2})/m(S_{1/2}) = m_\mu/m_e$. We then get: $l = 4{,}896 \cdot 10^{-91}\, \hbar/m_p c$, $M = 15{,}32\, m_p$. Hence the deviation from Maxwell's electrodynamic is extremely small, but not zero, and heavy leptons should exist near $m = |M|$. Some further leptonic states exist with masses similar to that of the muon. All states, those of the electron and the muon excepted, are γ-instable (life time 10^{-17} sec. resp. 10^{-26} sec.).

1. Introduction

Two mathematical equivalent methods exist in classical physics to describe the motion of point-charges in their own electromagnetic field, the field-theoretical one and the electromechanical one. In general, quantization is based on the fieldtheoretical aspect, and quantumelectrodynamics as the result is well known.

Here we consider the quantization of electro-mechanics. In particular, we study the motion of one pointcharge in its own electromagnetic field. The interaction integral was given by FOKKER [1] in 1929:

$$A = \tfrac{1}{2}\, \alpha \int G\big(x(\tau_1) - x(\tau_2)\big)\, \dot{x}^\mu(\tau_1)\; \dot{x}_\mu(\tau_2)\; d\tau_1\, d\tau_2\,. \qquad (1.1)$$

$G(x)$ is the Green function. That of Maxwell's electrodynamics has been written by DIRAC [2] in 1938 as

$$G(x) = \delta(x^2)\,. \qquad (1.2)$$

The Sommerfeld fine structure constant α enters this classical expression because we use $\hbar$ and c as units. In the last forties [3] we have investigated the same expression with generalized Green functions and obtained encouraging mass spectra. This approach was dropped in those days for several reasons, mainly because half odd spins were not available, and because the renormalization methods of quantum electrodynamics proved very successful. It is certainly a good luck that the mass problem may be eliminated for many essential problems, but, it would be unfortunate to resign completely. Therefore we came back to electromechanics [4], and now we have found a particular Green function with extraordinary simple properties. In particular, we obtain only a finite number of degrees of freedom and, by the way, half odd spins.

The Green function (1.2) is different from zero only on the light cone. Here, we shift the interaction surface without any smearing out. Hence we replace (1.2) by [5]

$$G(x) = \delta(x^2 + l^2) \qquad (1.3)$$

(metric: $-+++$). The shifting length l is unknown. But it will turn out that l cannot be zero, and that it is extremely small [6]:

$$l = 4.896 \times 10^{-91}\, \hbar/m_p c\,. \qquad (1.4)$$

It is hopeless to find any appropriate length measurement. But this small deviation from Maxwells theory is essential for the resulting mass spectrum.

As in quantum electrodynamics we do not obtain masses without assuming a bare mass M. That means: Masses come from outside. Electrodynamics, determined by (1.3), is only able to shift the masses. But this mass shifting may be appreciable.

At first, the bare mass M is unknown. However, as in the Dirac theory of the free electron we ob-

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* Presented as a *summar* to the Bayerische Akademie der Wissenschaften in the session of October 3rd, 1972, and as a letter to P. JORDAN on occasion of his 70th birthday, October 18th, 1972. Large numbers like $1/M = 10^{89}$ as eigenvalues of differential equations which contain only the fine structure constant α, may be of particular interest for the Dirac-Jordan theory of time depending gravitation.

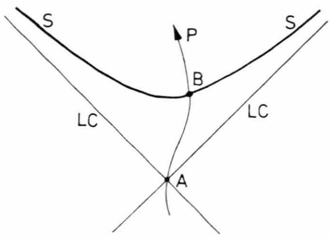

Fig. 1-1. Interaction surface S within the future part of the light cone LC and a pointcharge path P crossing the light cone in A and the interaction surface in B.

tain both signs $\pm M$. We start, so to say, with a "bare" and an "antibare". The result of this paper will be:

$$|M| = 15.32\, m_{\mathrm{p}}\,. \qquad (1.5)$$

Hence we expect leptons, dressed by photons. The masses ly, in general, nar the bare mass, however sometimes much lower.

Starting from (1.3) we obtain a Hamiltonian. The number of degrees of freedom is twice that of a Newtonian masspoint. That corresponds to the intersections A in B in Fig. 1-1 of the spacetime path P with the lightcone LC and with the interaction surface S. The Hamiltonian has a squareroot term, which can be linearized with Dirac matrices. Half odd spins result therefore as in the Dirac theory of free electrons. Whether half odd spins exist or not, depends essentially on the choice of the Green function. If, for example, we replace (1.3) by

$$G(x) = \delta(x^2 + l^2) + \gamma\, l^2\, \delta'(x^2 + l^2) \qquad (1.6)$$

we obtain integer spin values, although both expressions, (1.3) and (1.6), have the same limit (1.2) if $l \to 0$. Therefore it is necessary to change (1.2). Without changing the Green function we do not know really which limiting process is meant [7].

Once we have linearized the Hamiltonian we obtain a four component wave equation similar to the Dirac equation of a relativistic H-atom if we consider only solutions of the momentum $\boldsymbol{P} = 0$. In this case the eigenvalues are rest masses, and the wave-equation depends only on three internal coordinates given by the vector AB in Figure 1-1. The angles of this vector may be separated. We obtain the angular momentum quantum numbers

$$k = \pm 1, \pm 2, \pm 3 \ldots \qquad (1.7)$$

and the respective states

$$\begin{array}{llll} M = -|M|: & \mathrm{S}_{1/2} & \mathrm{P}_{3/2} & \mathrm{D}_{5/2} \ \text{etc.} \\ M = +|M|: & \mathrm{P}_{1/2} & \mathrm{D}_{3/2} & \mathrm{F}_{5/2} \end{array} \qquad (1.8)$$

The lowest states with the angular momenta $k > 0$ (first line) belong to the "antibare", the other ones with $k < 0$ (second line) to the "bare".

If we identify the lowest states $\mathrm{S}_{1/2}$ and $\mathrm{P}_{1/2}$ respectively with the electron and the muon, we obtain by numerical integration the above given values for l and M, and, in relative units, the following lowest mass values $m(k)$ for the different k's:

$$\begin{aligned} m(+1) &= 1\,, \\ m(-1) = m(+2) &= 206.3\,, \\ m(-2) = m(+3) &= 305.3\,, \\ m(-3) = m(+4) &= 377.1\,. \end{aligned} \qquad (1.9)$$

Before discussing these results we shall develop the theory. Up to the numerical integration of the two-component radial wave equation all calculations are exact.

2. The Hamiltonian

We start with the interaction integral (1.1) and with the Green function (1.3). Using $\hbar$, c and l as units we obtain

$$A = \tfrac{1}{2}\, a \int \delta([x(\tau_1) - x(\tau_2)]^2 + 1)\, \dot{x}^\mu(\tau_1) \cdot \dot{x}_\mu(\tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\,. \qquad (2.1)$$

Introducing $\tau_1 = \tau + \lambda$, $\tau_2 = \tau - \lambda$, and

$$x_1(\tau) := x(\tau + \lambda), \quad x_2(\tau) := x(\tau - \lambda) \qquad (2.2)$$

the λ-integration can be done easily [8]. Introducing

$$X = \tfrac{1}{2}(x_1 + x_2), \quad x = x_1 - x_2 \qquad (2.3)$$

as external and internal coordinates in the space-time we get the Lagrangean

$$L = -\frac{a}{2}\, \frac{\dot{X}^2 - \frac{1}{4}\dot{x}^2}{\dot{X} \cdot x}\,, \quad x^2 = -1\,. \qquad (2.4)$$

The denominator is similar the that of the Liénard-Wiechert-potentials [9]. Seperating space- and time-coordinates by

$$\begin{aligned} &X = (t, \boldsymbol{R}), \quad x = (\sqrt{1+r^2}, \boldsymbol{r}), \quad \tau = t, \quad r = |\boldsymbol{r}|, \\ &\dot{X} = (1, \dot{\boldsymbol{R}}), \quad \dot{x} = \left(\frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{\sqrt{1+r^2}}, \dot{\boldsymbol{r}}\right), \end{aligned} \qquad (2.5)$$

(2.4) can be written as

$$\begin{aligned} L &= -\frac{a}{2}\, \frac{1 - \dot{\boldsymbol{R}}^2 + Q}{\sqrt{1+r^2} - \dot{\boldsymbol{R}} \cdot \boldsymbol{r}}\,, \\ Q &= \frac{1}{4}\left(\dot{\boldsymbol{r}}^2 - \frac{(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})^2}{1+r^2}\right). \end{aligned} \qquad (2.6)$$

As it is well known, we may introduce particular integrals of motion into the Lagrangean. Since the momentum

$$\boldsymbol{P} = \frac{\partial L}{\partial \dot{\boldsymbol{R}}} = \frac{a \dot{\boldsymbol{R}}}{N} - \frac{a Z}{2 N^2} \boldsymbol{r},$$

$$Z = 1 - \dot{\boldsymbol{R}}^2 + Q, \quad N = \sqrt{1 + r^2} - \dot{\boldsymbol{R}} \cdot \boldsymbol{r}, \qquad (2.7)$$

is an integral of motion, we may consider only solutions with the external momentum $\boldsymbol{P} = 0$. Hence

$$\dot{\boldsymbol{R}} = (Z/2N)\, \boldsymbol{r}, \qquad (2.8)$$

and it follows by straight forward calculations:

$$L = \mp (a/r^2) \sqrt{1 - r^2 Q} \mp (a/r^2) \sqrt{1 + r^2}, \qquad (2.9)$$

i. e. the Lagrangean for the internal motion of the vector $\overrightarrow{AB}$ in Fig. 1-1. We get two independent double-signs.

Now we derive the corresponding Hamiltonian. The internal momentum equals

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = \pm \frac{a}{2\sqrt{1 - r^2 Q}} \frac{\partial Q}{\partial \dot{\boldsymbol{r}}}.$$

According to (2.6)

$$\dot{\boldsymbol{r}} \cdot \partial Q / \partial \dot{\boldsymbol{r}} = 2 Q$$

we get as energy

$$H = \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - L = (a/r^2) (\pm \sqrt{1 + r^2} \pm 1/\sqrt{1 - r^2 Q}). \qquad (2.10)$$

Inserting the explicite expression for $\boldsymbol{p}$ we obtain

$$\boldsymbol{p}^2 + (\boldsymbol{p} \cdot \boldsymbol{r})^2 = \frac{a^2 Q}{4(1 - r^2 Q)} \qquad (2.11)$$

and, hence, the Hamiltonian

$$H = \pm (1/r^2) \sqrt{a^2 + 4 r^2 (\boldsymbol{p}^2 + (\boldsymbol{p} \cdot \boldsymbol{r})^2)} \pm (a/r^2) \sqrt{1 + r^2}. \qquad (2.12)$$

According to $\boldsymbol{P} = 0$ that Hamiltonian represents the rest mass:

$$m = H. \qquad (2.13)$$

3. The Wave Equation for Masses

The left side of Eq, (2.11) may be written as a square of a vector:

$$\boldsymbol{p}^2 + (\boldsymbol{p} \cdot \boldsymbol{r})^2 = (\boldsymbol{p} + (1/r^2)(\sqrt{1 + r^2} - 1)(\boldsymbol{p} \cdot \boldsymbol{r})\, \boldsymbol{r})^2.$$

Using the Dirac matrices $\boldsymbol{\varrho} = \boldsymbol{\sigma}^{\mathrm{P}} \times 1$, $\boldsymbol{\sigma} = 1 \times \boldsymbol{\sigma}^{\mathrm{P}}$ ($\boldsymbol{\sigma}^{\mathrm{P}}$ = 2×2-Pauli-matrices), and inserting $\boldsymbol{p} = -\mathrm{i} \nabla$ we obtain the generalized Schrödinger-Operator (= Hamiltonian in the Schrödinger picture)

$$H = -\frac{2 i \varrho_1 \boldsymbol{\sigma}}{r} \left(\nabla + \frac{1}{r^2} (\sqrt{1 + r^2} - 1)\, \boldsymbol{r} (\boldsymbol{r} \cdot \nabla) \right) + \frac{i \varrho_1 \boldsymbol{\sigma} \cdot \boldsymbol{r}}{r^3} \left(2 - 2\sqrt{1 + r^2} + \frac{1}{\sqrt{1 + r^2}} \right) + \frac{a}{r^2} \sqrt{1 + r^2} + \frac{a}{r^2} \varrho_3 + M \varrho_3. \qquad (3.1)$$

The first term in the second line makes the expression in the first line Hermitean. The last second line term is added as the contribution of the bare mass. The double-sign of the first square root in (2.12) is included in the matrix notation as in the Dirac case. We obtain the well known positive and negative eigenvalues. The second double-sign in (2.12) may be shifted to the bare mass which equals either $M = +|M|$ or $-|M|$. This sign will be responsible for the existence of two γ-stable particles, the electron, and the muon. One of them is, so to say, a dressed "bare", the other one a dressed "antibare". This assumption takes into account the experience, that both particles are connected with a non quantum electrodynamic quantum number, and it turns out that this number will be deeply woven into the frame work of quantum electromechanics.

As in the theory of the relativistic H-atom [10] we separate the angular momentum part. Using

$$\sigma_r := \varrho_1 \boldsymbol{\sigma} \cdot \boldsymbol{e}, \quad \boldsymbol{r} = r\, \boldsymbol{e}, \quad \boldsymbol{e}^2 = 1, \qquad (3.2)$$

we obtain

$$H = -\frac{2 i \varrho_1 \sigma_r}{r} \left((\varrho_1 \boldsymbol{\sigma} \cdot \boldsymbol{e})(\varrho_1 \boldsymbol{\sigma} \cdot \nabla) + (\sqrt{1 + r^2} - 1) \frac{\partial}{\partial r} \right) + \frac{i \varrho_1 \sigma_r}{r^2} \left(2 - 2\sqrt{1 + r^2} + \frac{1}{\sqrt{1 + r^2}} \right) + \frac{a}{r^2} \sqrt{1 + r^2} + \left(M + \frac{a}{r^2} \right) \varrho_3.$$

According to

$$(\varrho_1 \boldsymbol{\sigma}\cdot\boldsymbol{e})(\varrho_1 \boldsymbol{\sigma}\cdot\nabla) = \frac{\partial}{\partial r} - \frac{\varrho_3}{r} K + \frac{1}{\sqrt{1+r^2}}, \quad K = \varrho_3(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1), \quad \boldsymbol{L} = -i\,\boldsymbol{r}\times\nabla, \tag{3.3}$$

taking into account that K is an integral of motion with the eigenvalues $k = \pm 1, \pm 2$ etc. which corresponds to the states (1.8), – replacing the remaining matrices $\varrho_1 \sigma_1$ and ϱ_3 by the algebraicly equivalent ones ϱ_2 and ϱ_3, we obtain finally:

$$\begin{aligned} H = &-\frac{2i\varrho_2}{r}\left(\sqrt{1+r^2}\frac{d}{dr} - \frac{k}{r}\varrho_3 + \frac{1}{r}\right) \\ &+ \frac{i\varrho_2}{r^2}\left(2 - 2\sqrt{1+r^2} + \frac{1}{\sqrt{1+r^2}}\right) + \frac{a}{r^2}\sqrt{1+r^2} + \left(M + \frac{a}{r^2}\right)\varrho_3 . \end{aligned} \tag{3.4}$$

According to the algebraic meaning, ϱ_2 and ϱ_3 may be represented by the respective 2×2-Pauli-matrices. Hence we get the following 2-component relativistic wave equation for the masses of a pointcharge

$$\begin{aligned} -\frac{2}{r}\left(\sqrt{1+r^2}\frac{d}{dr} + \frac{k}{r} + \frac{1}{r}\sqrt{1+r^2} - \frac{1}{2r\sqrt{1+r^2}}\right)\psi_2 &= \left(m - M - \frac{a}{r^2}\sqrt{1+r^2} - \frac{a}{r^2}\right)\psi_1 , \\ +\frac{2}{r}\left(\sqrt{1+r^2}\frac{d}{dr} - \frac{k}{r} + \frac{1}{r}\sqrt{1+r^2} - \frac{1}{2r\sqrt{1+r^2}}\right)\psi_1 &= \left(m + M - \frac{a}{r^2}\sqrt{1+r^2} + \frac{a}{r^2}\right)\psi_2 . \end{aligned} \tag{3.5}$$

Inserting

$$t = \sqrt{1+r^2}, \quad \frac{d}{dr} = \frac{r}{\sqrt{1+r^2}}\frac{d}{dt} = \frac{\sqrt{t^2-1}}{t}\frac{d}{dt} \tag{3.6}$$

we obtain

$$\begin{aligned} \left(\frac{d}{dt} + \frac{k}{t^2-1} + \frac{t}{t^2-1} - \frac{1}{2t(t^2-1)}\right)\psi_2 &= \frac{1}{2}\left(M - m - \frac{a}{t-1}\right)\psi_1 , \\ \left(\frac{d}{dt} - \frac{k}{t^2-1} + \frac{t}{t^2-1} - \frac{1}{2t(t^2-1)}\right)\psi_1 &= \frac{1}{2}\left(M + m + \frac{a}{t-1}\right)\psi_2 . \end{aligned} \tag{3.7}$$

A remarkable simplification occurs if we put

$$\psi_1 = \frac{1}{\sqrt[4]{t^2(t^2-1)}}\sqrt{\frac{t-1}{t+1}}^{\,k} u, \quad \psi_2 = \frac{1}{\sqrt[4]{t^2(t^2-1)}}\sqrt{\frac{t+1}{t-1}}^{\,k} v . \tag{3.8}$$

So we obtain

$$\frac{du}{dt} = \frac{1}{2}\left(\frac{t+1}{t-1}\right)^k \frac{(M+m)\,t + (M+m-a)}{t+1}\,v , \qquad \frac{dv}{dt} = \frac{1}{2}\left(\frac{t-1}{t+1}\right)^k \frac{(M-m)\,t - (M-m-a)}{t-1}\,u . \tag{3.9}$$

In the asymptotic case $t\to\infty$ we have

$$du/dt = \tfrac{1}{2}(M+m)\,v , \qquad dv/dt = \tfrac{1}{2}(M-m)\,u . \tag{3.10}$$

Hence

$$d^2u/dt^2 = \tfrac{1}{4}(M^2-m^2)\,u , \qquad d^2v/dt^2 = \tfrac{1}{4}(M^2-m^2)\,v ,$$

and

$$u \to \sqrt{M+m}\,e^{-\gamma t}, \qquad v \to \sqrt{M-m}\,e^{-\gamma t}, \qquad \gamma = \tfrac{1}{2}\sqrt{M^2-m^2} . \tag{3.11}$$

Therefore eigensolutions are only possible if

$$m^2 < M^2 . \tag{3.12}$$

Obviously the bare mass must be different from zero if eigensolutions shall exist.

In the Eqs. (3.9) a is a well known constant, the masses m are defined as eigenvalues, but M and l are still to be determined. The length l will be scaled by identifying the lowest mass m with the experimental electron mass m_e. The bare mass M is hypothetically choosen so that the ratio of the two lowest eigenvalues m becomes equal to the experimental ratio m_μ/m_e. Then we may expect that M is not too far away from the value we obtain if $m = 0$ and

M is an eigenvalue of the remaining equation (3.9) for $k=+1$:

$$\frac{du}{dt} = \frac{1}{2}\left(M - \frac{a-2M}{t-1}\right) v\,,$$
$$\frac{dv}{dt} = \frac{1}{2}\left(M + \frac{a-2M}{t+1}\right) u\,. \tag{3.13}$$

We insert $t=1+x$ and replace M by $-M$, that means, bares by antibares, and obtain

$$\frac{du}{dx} = -\frac{1}{2}\left(M + \frac{a+2M}{x}\right) v\,,$$
$$\frac{dv}{dx} = -\frac{1}{2}\left(M - \frac{a+2M}{x+2}\right) u\,. \tag{3.14}$$

The solutions should be regular within $0 \leqq x \leqq \infty$. Obviously, $v \sim x$ near $x=0$, and it has a maximum at $x=a/M$. According to (3.11) it yields asympotically

$$u,\, v \sim e^{-M/2}\,. \tag{3.15}$$

If we eliminate v in (3.14), we obtain

$$u'' + \left(\frac{1}{x} - \frac{M}{Mx+a+2M}\right) u' - \left(\frac{1}{4}\,\frac{(Mx+a+2M)(M+a)}{x(x+2)}\right) u = 0\,. \tag{3.16}$$

Hence we have poles at $x=-2-a/M$, $x=-2$, $x=0$ and an essential singularity at $x \to \infty$. Since we need a regular solution in the intervall $(0, \infty)$, and since we obtain at $x=-2$ as characteristic equation $\nu^2=0$, the solutions u and v must be logarithmically singular at $x=-2$. Therefore we make the approximative ansatz:

$$v = \ln\frac{x+2}{2}\, e^{-Mx/2}\,,$$
$$v' = \left(\frac{1}{x+2} - \frac{M}{2}\ln\frac{x+2}{2}\right) e^{-Mx/2}\,. \tag{3.17}$$

Assuming that this rough approximation provides us with the right position of the maximum at $x=a/M$ we get

$$\ln(\xi+1) = \frac{2}{a}\,\frac{\xi}{\xi+1}\,, \qquad \xi = \frac{a}{2M}\,,$$

and $$\xi = e^{2/a}\,, \quad M = \frac{a}{2}\, e^{-2/a}\,.$$

Here M is much smaller than the value $M \approx e^{-3/2\,a}$ computed in § 4. So far the approximation is very poor. However, it makes clear, how extraordinarily small eigenvalues may occur. It is not yet a proof, but a test, which may give some further confidence into the numerical results in § 4.

4. Numerical Solutions

Analytical methods seem to be not available for solving (3.9). Since we have no eigensolutions at all if $M=0$, perturbation methods are excluded although M will be extremely small. Variational methods are not useful, because it turns out that eigenvalues will exist near $m=-M$. That is far below the domains $|m| \ll |M|$ in which we are mainly interested. Methods like that indicated after Eq. (3.17), may give some, but only qualitative informations. Therefore, we use computer methods and solve (3.9) with appropriate initial values, trying to obtain mass values which give the right asymptotic behaviour. Even that is not quite easy, because tremendous intervals must be taken into account.

Writing (3.9) with $t=x+1$ we obtain

$$\frac{du}{dx} = \frac{1}{2}\left(\frac{x+2}{x}\right)^k \left(M+m-\frac{a}{x+2}\right) v\,,$$
$$\frac{dv}{dx} = \frac{1}{2}\left(\frac{x}{x+2}\right)^k \left(M-m+\frac{a}{x}\right) u\,. \tag{4.1}$$

Typical features appear if we insert $k=+1$, $M=-|M| =: -M'$:

$$\frac{du}{dx} = -\frac{1}{2}\,\frac{(M'-m)(x+2)+a}{x}\, v =: -f(x)\, v\,,$$
$$\frac{dv}{dx} = -\frac{1}{2}\,\frac{(M'+m)\,x-a}{x+2}\, u =: -g(x)\,(x-\alpha)\, u\,, \tag{4.2}$$

where $f(x)>0$, $g(x)>0$, $\alpha = a/(M'+m)$. Hence we have oscillating solutions (or at least such concave versus the abszissa) for $x<\alpha$ and divergent ones if $uv<0$ for $x \geqq \alpha$. Therefore the step by step calculations are to be stopped at $x=\alpha$ if $uv<0$. In the other cases we may stop the calculationes beyond $x=\alpha$ at a knot of u or v.

We have used the computer of the Leibniz-Rechenzentrum at Munich. The rather fast program library procedure "Diffsys", appropriately used, proved to be as reliable as a Runge-Kutta method, which provides us with essentially the same results. The step-width is automatically adjusted until variations within a given intervall are less than a given fraction, eps, of the maximum.

Error propagation is calculated with the transportation matrix. It seems to be sufficient that eps $=10^{-10}$. Since (4.1) is singular at $x=0$, the initial values are calculated for $x=10^{-5}$ by iteration.

First tests of the procedure are made by varying eps, and the length of the intervals. Terms of the order of 10^{90} in some terms of (4.1) makes further tests advisable. To this end we have integrated some similar differential equations by Diffsys and analytically, namely for $x \leqq 10^{10}$:

$$\frac{dv}{dx} = -\frac{a}{2}\frac{(x+2)^{k+1}}{x^k}u\,,\quad \frac{du}{dx} = -\frac{a}{2}\frac{x^{k-1}}{(x+2)^k}v\,, \tag{4.3}$$

and for $10^{10} \leqq x \leqq 10^{70}$:

$$\frac{dv}{dx} = -\frac{a}{2}u\,,\quad \frac{dv}{dx} = +\frac{a}{2x}u\,. \tag{4.4}$$

Eq. (4.3) has been integrated within $0 \leqq x \leqq 10^{10}$ by Diffsys, and hence (4.4) within $10^{10} \leqq x \leqq x_0$ by Diffsys and within $x_0 \leqq x \leqq 10^{24}$ analytically. Varying x_0 from 10^{10} to 10^{70} the relative error is less than 10^{-6}. Taking further into account that the obtained wavefunctions are very smoth, we guess that the error of results is less than 10%.

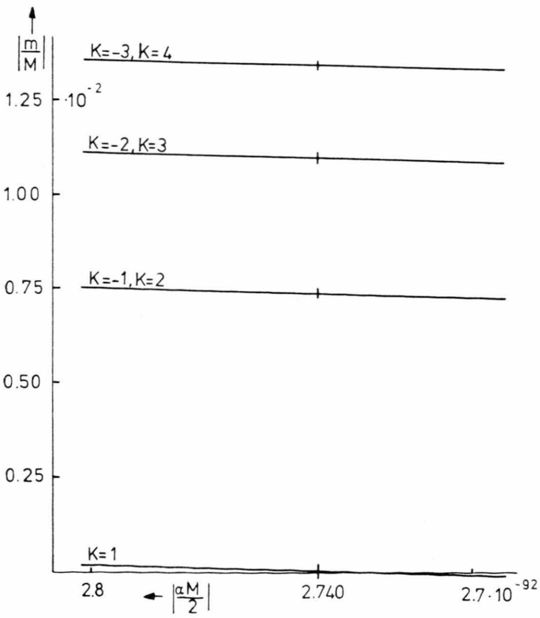

Fig. 4-1 b. Dependence of the lowest eigenvalues on M in the interesting interval of M.

The eigenvalues will depend on M. The variation with M is shown in Fig. 4-1a, b for the lowest states belonging to $k=+1, -1$, 3 and 4. According to

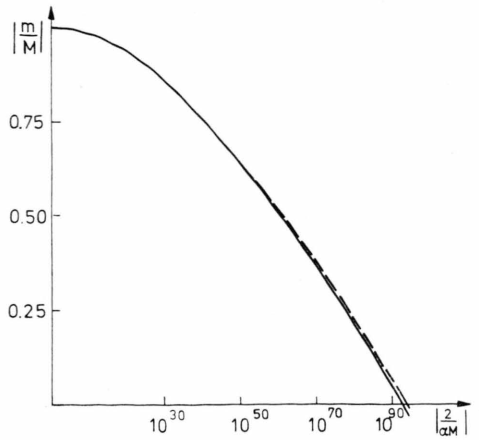

Fig. 4-1 a. Dependence of the lowest eigenvalues on M, the solid line is for $k=1$, the dashed one for $k=-1$.

Fig. 4-1 we may choose M so that the ratio of $m(+1)$ and $m(-1)$ equals that of the electron and the muon:

$$m(-1)/m(+1) = m_\mu/m_e\,.$$

That means: We identify the electron and the muon with the lowest spectral stets $S_{1/2}$ and $P_{1/2}$, which have opposite parities as it might be possible [11]. In this case M must be extraordinary small:

$$M = 7.50 \times 10^{-90}\,,\qquad l = 0.4896 \times 10^{-90}/m_p\,,$$

and with this value for M we obtain for the lowest states

$k>0$	State	$k<0$	State	$-m/M$	m/m_e	m MeV
+1	$S_{1/2}$	—	—	3.555×10^{-5}	1	0.511
+2	$P_{3/2}$	−1	$P_{1/2}$	7.335×10^{-3}	206.3	105.4
+3	$D_{5/2}$	−2	$D_{5/2}$	10.855×10^{-3}	305.3	156.0
+4	$F_{7/2}$	−3	$F_{7/2}$	13.405×10^{-3}	377.1	192.7

$M=-|M|$ $M=+|M|$; $-\ln|M|=205.6$ $|M|/m_\mu=136.32$;

Of course, the bare mass cannot be observed. However, the excited states of $S_{1/2}$, $P_{1/2}$ etc. differ only very few from the bare mass:

$$(M-m)/M \approx 10^{-4}\,.$$

They are nearly the same within the range of some electron masses.

Masses, belonging to the same "orbital" momentum l, are nearly equal. It is not a strict degeneracy. However, if we assume $M=0$, and neglect m against

M, Eq. (4.1) will be invariant under the transformation

$$k \to 1-k, \quad u \to v, \quad v \to -u .$$

According to the small values of M and m we have a weak breaking of this symmetry.

The eigenfunctions are characterized by a fast increase (length: $1/|M| \approx 1 \times 10^{-104}$ cm), and a slow decrease (length: $1/|M| \approx 1.4 \times 10^{-15}$ cm for electrons and particles with muonic masses, $1/\sqrt{M^2-m^2} \approx 10^{-13}$ cm for states with masses near the bare one). Hence, the particles near the bare mass have nuclear dimensions, and the electron and the muonic particles should be about ten times smaller than the Compton wave length ($h/m_p c = 2.1 \times 10^{-14}$ cm) of the proton. The extension increases with decreasing mass according to $1/\sqrt{M^2-m^2}$.

5. Some Concluding Considerations

We have fitted the constants M and l by the mass of the electron and by the mass ratio m_μ/m_e. The first fit tells us only, what l is if measured in conventional length units. The second one includes the hypothesis that the muon is an electromechanical state. Nevertheless, the fact that the muon may be incorporated is, as far as we know, a new result.

If we accept this hypothesis, both particles, the electron and the muon, have some stimulating quantum numbers. The electron is a $S_{1/2}$-state and the muon a $P_{1/2}$ one. In so far we may consider the mass difference between electrons and muons as a kind of Lamb-shift. However, the electron belongs to the bare mass $M = -|M|$ and the muon to $M = +|M|$. Hence, the particles differ in the bare-anti-bare-quantum number which is connected with the non-electromagnetic mass $|M|$.

What about the other quantum states? Heavy leptons as well as those with nuonic masses and higher spins are unknown. One of the reasons for that may be the γ-instability. The dipole lifetimes is given by

$$\tau = 3/2\, \alpha \cdot (c/a\,\omega)^2/\omega$$

($\omega = 2\pi$-transition-frequency, a = oscillator amplitude). We have in common units:

$$a = 2\,\hbar/c\,\sqrt{M^2-m^2}, \qquad \omega = c^2\,\Delta m/\hbar ,$$

and

$$\tau = 3/2\,\alpha \cdot (\sqrt{M^2-m^2}/2\,\Delta m)^2 \cdot \hbar/c^2\,\Delta m .$$

Hence, we obtain roughly ($2\pi\Lambda$ = Compton-wave-length; say, Λ = Compton-length):

leptons	a	c/ω	τ (sec)
heavy	$4\,\Lambda_p$	$\Lambda_p/15$	0.27×10^{-26}
muonic	$2\,\Lambda_p/15$	Λ_μ	0.24×10^{-17}

(5.1)

Hence, all states are γ-instable excepted the lowest ones with different signs of the bare mass $M = \pm|M|$, that means, only those which has been identified above with the electron and the muon.

The other states with muonic masses might be observed as resonances, e. g.

$$\gamma + e \to (P_{3/2}) \to \gamma + e . \tag{5.2}$$

The resonances with a heavy leptons as an intermediate state are probably unobservable because the expected line width is larger than the frequency. Even in (5.2) the cross section may be smaller than that of the Compton effect by some powers of $m_e/m\,(+2)$.

In addition, there may exist further reasons, others than the γ-instability, which explain that the γ-instable states are not yet observed. The probabilities for the pair production of heavy leptons are extremely small, as it has been discussed in the context of pair production of nuclei heavier than the triton [12].

The pair production of particles with muonic masses and "orbital" momenta $l \geqq 2$ should be improbable relative to those with $l \leqq 1$ because the distances, at which pair production may occur, must be much larger, and, hence, the interaction much weaker.

Therefore only the pair production of the state $P_{3/2}$ might compete with that of the muon. Hence, we look for the process

$$\begin{aligned} \text{Nucl} + \gamma &\to \text{Nucl} + (P_{3/2}) + (P_{3/2}) \\ &\to \text{Nucl} + e^- + e^+ + 2\gamma . \end{aligned} \tag{5.3}$$

However, even this process may be remarkably less probable than that of the pair production of muons. For the spin states of a muon-, and of a $P_{3/2}$-pair are $s = 1$ or 0, respective $s = 3$, 2, 1 or 0, while only the states $s = 1$ are compatible with the spin $s = 1$ of the pair producing γ-quantum.

Nothing is known if there may exist some other production processes of particles with muonic masses at very high energies.

Summarizing we may state: There are two and only two γ-stable charged leptons, and probably some resonances in the muonic mass region. It seems to be rather probable that we have obtained a theory, which includes the real electron and the real muon. Certainly, we would be more happy if we were able to calculate their mass ratio. Howeve, we must take it as a fact that we obtain two new constants, the deviation length l and M, which appears here as a bare mass. The bare mass of the electron is negative and large as in quantum electrodynamics, but no longer infinite. The bare mass belonging to the muon is positive. The change of the sign of the bare masses warrants the γ-stability of both, the electron and the muon. Their existence seems to be well ebstablished.

Even if we consider the classical theory we can understand that we need at least two length constants. According to the Maxwell theory a point-charge will explode. To avoid this explosion we have, in principle, two types of hypotheses. We need either nonelectromagnetic forces, or a change of Maxwell's equation. If we require that the cohesive forces have the same symmetry as the fields in Maxwell's equations we need a modified connection between the fourvector A^μ and the current density J^μ:

$$A^\mu(x) = -\frac{\mu_0 e}{4\pi} \int G(x-x')\, J^\mu(x')\, d^4x', \quad (5.4)$$

where $G(x)$ is the generalized Green function. The G's may be characterized by the static potential:

$$\Phi(\mathbf{r}) = c \int G(+\mathbf{r}, -c\,t')\, dt'. \quad (5.5)$$

We obtain from $G = \delta(x^2)$ the Coulomb potential

$$\Phi(r) = 1/r, \quad (5.6)$$

and from $G = \delta(x^2 + l^2)$ the potential

$$\Phi(\mathbf{r}) = 1/\sqrt{r^2 + l^2}. \quad (5.7)$$

The latter one is finite at $r = 0$. But charges will still explode because $\Phi(r)$ is completely decreasing. Neither graph (a) in Fig. 5-1 corresponding (5.6), nor graph (b) corresponding (5.7) yields stable solutions. We need some potential of the kind given in graph (c) [13]. Hence, two constants are necessary at least for stable solutions. One of them may be represented by a bare mass. This means essentially that the increasing part of graph (c) is shrinked to something like a δ-function. We have seen that this simple classical result persists in quantum electromechanics.

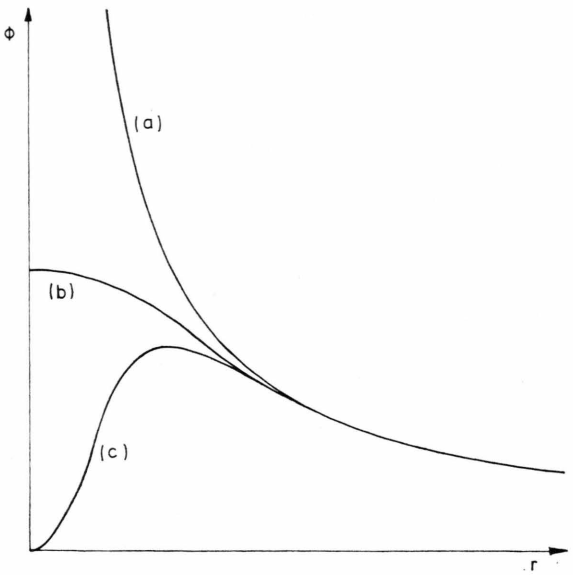

Fig. 5-1. Graphs for different static potentials, (a) Coulomb potential, (b) 1-constant-theory with a classically finite selfenergy, (c) 2-constant-theory with a classically vanishing selfenergy.

Changing Green's function is not en vôgue. The main reason against this procedure may be its arbitrariness. However, if the postulate of the persistence of the Eq. (5.4) holds, it is impossible to avoid it. The fear of arbitrariness may be transformed into the question, how the particular change may be understood.

The tremendously small value of the deviation length l is responsible for the square integrability of the wave function, and for its rapid increase near $x = 0$. Taking into account all we know on physics today, only gravity may explain such a small length. Once more we have an indication that gravity may be responsible for finite results. However we must keep in mind that completely new effects cannot yet be excluded. But we must also take into account that $(G\, m_e^2/\hbar\, c)^2 \approx 3 \times 10^{-90}$.

The bare mass M defines the rather slow decrease of the wave function. It is essentially responsible for the magnitude of the masses, and for the extension of the particles. It seems to be encouraging that the heavy leptons have an extension similar to that of the proton and the neutron.

Taking into account the results on the numerical values of l and M, one may feel that some propositions should exist to derive a Green function with two characteristic constants, perhaps just that given here. It is an experimental fact that the fine structure constant may be much more important than it is believed today. We remind (i) that the so called classical electron radius which is no longer the electron radius but only a characteristic electronic constant, equals nearly the Compton-length $\Lambda = \hbar / m_\pi c$ of the pion, (ii) that the muon mass equals nearly $3/2\,\alpha$ in units m_e, (iii) that the ratio of the Compton-length of the proton and the universal fermi length $l_F = 0.7 \times 10^{-16}$ cm equals within 10% the ratio of the pion and the electron mass. If all these numerical results really indicate that α plays a certain rôle, the postulate of the persistence of the Maxwellian symmetry expressed by (5.4) and underlying the electromechanics, appears well established. – "It's a long way to Tipparary", but it is, hopefully, a promissing one.

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[6] The Coulomb-Potential $1/r$ is changed into $1/\sqrt{r^2+l^2}$. The deviation length l is much smaller than the electron's gravity length.

[7] We share, partially, the scepticism concerning ad hoc variations of the Green function. However, we see that the unchanged one is not well defined. Therefore it is necessary to study changes until we find axioms to derive them. The extremely small value of l may be a hint that the real change is due to a gravitational effect.

[8] F. BOPP and W. LUTZENBERGER, Acta Nova Leopoldina 1972, in press: Ein neues Näherungsverfahren zur Berechnung von Massenspektren. Lecture held at the Wartburg-Symposium 1972.

[10] Cf. e. g. A. SOMMERFELD, Atombau und Spektrallinien, Bd. II, 2. Aufl., F. Vieweg & Sohn, Braunschweig 1944, Kap. 4, § 7.

[11] As long as weak interaction is not taken into account.

[12] Cf. e. g. V. N. BAIER, V. M. KATKOV, and V. M. STRAKHOVENKO, Sov. Jour. Nucl. Phys. **14**, 572 [1972]. We are thanking H. LEHMANN for this hint.

[13] Even ROHRLICH's statement [14] may be included here that there is no selfenergy at all. In fact, the classical selfenergy vanishes if $\Phi(0) = 0$ as in Fig. 5-1, graph (c), and the ROHRLICH-assumption results if the maximum of graph (c) is shifted to $r \to 0$, $\Phi \to \infty$ in such a manner that $\Phi(0) = 0$ is not changed and $\Phi(r)$ remains below graph (a). However, we may obtain non-vanishing eigenvalues m after quantization even if $\Phi(0) = 0$.

[14] F. ROHRLICH, Proceedings of the Trieste Symposium of the Development of the Physicist's Conception of Nature, 1972, in press.

[9] Cf. e. g. A. SOMMERFELD, Vorl. Theoretische Physik, Bd. III, Elektrodynamik, 3. Aufl., AVG-Leipzig, § 30.